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STEADY MOTION OF A VISCOUS CONDUCTING FLUID IN RECTANGULAR PIPES UNDER THE INFLUENCE OF A TRANSVERSE EXTERNAL MAGNETIC FIELD

G. A. Grinberg

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STEADY MOTION OF A VISCOUS CONDUCTING FLUID IN RECTANGULAR PIPES UNDER THE INFLUENCE OF A TRANSVERSE EXTERNAL MAGNETIC FIELD /1721*

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ABSTRACT

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The author discusses flow in a pipe of arbitrary cross section with nonconducting walls. If the velocity v and induced magnetic field H are parallel to the axis of the pipe and the external field is uniform, the solution of the problem essentially involves determination of the Green's function for the two-dimensional equation $\Delta\psi - M^2\psi = 0$ and the corresponding region (M is the Hartmann number). The author has written an integral equation for the derivative of the Green's function normal to the contour, through which the entire solution is expressed in quadratures. For the case of large M, Grinberg has presented a rapidly converging process of successive approximations and obtained an asymptotic solution of the integral equation, by means of which it is possible to find the values v and H at any point within the pipe. The case of a rectangular pipe, for which the Green's function is determined precisely, also is considered.

Author

*Numbers given in the margin indicate the pagination in the original foreign text.

1. It has been demonstrated by Shercliff in ref. 1 that if an external magnetic field H^0 is uniform and the field of velocities and the induced electric and magnetic fields are not dependent on the z-coordinate, read in the direction of the axis of the pipe, there is a solution of the equations of steady motion of a conductive viscous incompressible fluid along the pipe when the velocity v and the induced magnetic field H are directed parallel to the z-axis and satisfy the equations

$$\Delta H + \frac{4\pi\mu\sigma H^0}{c^2}\frac{\partial v}{\partial x} = 0, \tag{1.1}$$

$$\Delta v + \frac{\mu H^0}{4\pi\eta}\frac{\partial H}{\partial x} = -\frac{P^0}{\eta}, \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{1.2}$$

here the x-axis was selected in the direction of the field H^0; σ, μ, η, and c denote conductivity, magnetic permeability, the coefficient of viscosity of the fluid and the speed of light respectively; a $-\frac{\partial p}{\partial z} = P^0$ is the pressure gradient, assumed to be constant in the cross section of the pipe.

The boundary conditions on the walls of the pipe, which we will consider nonconducting, cause the two values v and H to become equal to zero on the contour s of the pipe.

The solution of the problem has been given in ref. 1 for a pipe with a rectangular cross section and in ref. 4 for a round pipe. In both cases, the solution is presented in the form of trigonometric series obtained by the particular solutions method. Such a form of solution, suitable for small values of the Hartmann number, is very unsuitable in the case of its large values, since the convergence of the series worsens badly with an increase of the Hartmann number. The situation here is similar to that which occurs in the theory of wave diffraction on bodies of finite size, where the convergence of the

series obtained using the particular solutions method worsens sharply with an increase of the ratio of the characteristic size of a body to wavelength. For this reason it is necessary either to transform the derived series to a different form under these conditions, in order to obtain a form of solution more suitable for large values of the Hartmann number (for a rectangular pipe the appropriate procedure already has been noted in Shercliff's first study, ref. 1), or seek some other approach to a precise solution of the problem.[1] Such /1722
an approach, applicable to the case of a rectangular pipe, was presented in our paper cited as ref. 5 and involves the use of the corresponding Green's function for an equation of the form $\Delta u - m^2 u = 0$, where

$$m = \frac{\mu H^0}{2c} \sqrt{\frac{\sigma}{\eta}}.$$

This paper will present a generalization of this method for the case of a pipe of arbitrary cross section. Particular attention is given to the case of large Hartmann numbers. As will be shown below, it is especially in the case of large Hartmann numbers that this method is particularly effective and makes it possible to obtain an approximate solution of the problem in very simple form.

2. Proceeding to the solution of the problem, we introduce new functions in accordance with the usual procedure, specifically

$$w_{\pm} = v \pm (\alpha H + qx), \quad \alpha = \frac{c}{4\pi \sqrt{\sigma\eta}}, \quad q = \frac{p_0}{2\eta m}, \tag{2.1}$$

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[1]An approximate solution of the problem, based on physical considerations, making it possible to simplify the initial equations, has been given in refs. 1-3, 6 and elsewhere.

for which equations of the following form are obtained

$$\Delta w_{\pm} \pm 2m \frac{\partial w_{\pm}}{\partial x} = 0 \tag{2.2}$$

with the boundary conditions

$$w_{\pm}|_s = \pm q x|_s \equiv \pm q \xi(s). \tag{2.3}$$

Assuming further

$$w_{\pm} = e^{\mp m x} \varphi_{\pm}, \tag{2.4}$$

we obtain for the function $\varphi_{\pm}$ the single-type equation

$$\Delta \varphi_{\pm} - m^2 \varphi_{\pm} = 0 \tag{2.5}$$

and the boundary conditions

$$\ldots q e^{\pm m\xi(s)} \xi(s). \tag{2.6}$$

Now introducing into consideration the Green's function for equation (2.5), having the form[1]

$$g(P, Q) \equiv g(x, y, \xi, \eta) = -\frac{1}{2\pi} K_0(mr) + u(x, y, \xi, \eta), \tag{2.7}$$

[1]This obviously is related directly to the corresponding Green's function for a two-dimensional Helmholtz equation $\Delta \varphi + k^2 \varphi = 0$, since equation (2.5) differs from a Helmholtz equation only by replacement of the real wave number k by the imaginary wave number, $k = im$.

where $r = PQ$ is the distance between the fixed point $P(x, y)$ and the variable point $Q(\xi, \eta)$ of the region bounded by the contour s; $K_0(z)$ is MacDonald's function; $u(x, y, \xi, \eta)$ is a solution of equation (2.5), not having singularities within s, satisfying on s the equation

$$u|_s = \frac{1}{2\pi} K_0(mr)|_s,$$

so that $g|_s = 0$, we obtain

$$\varphi_{\pm}(P) \equiv \varphi_{\pm}(x, y) = \int_{(s)} \varphi_{\pm}|_s \frac{\partial g}{\partial n} ds \equiv \int_{(s)} \varphi_{\pm}(Q) \frac{\partial g}{\partial n} ds, \qquad (2.8)$$

where n is the external normal to the contour s and $Q(\xi, \eta)$ is a point of the contour. This gives further

$$v = \frac{1}{2}(\varphi_+ e^{-mz} + \varphi_- e^{mz}) = -q \int_{(s)} \xi \operatorname{sh} m(x - \xi) \frac{\partial g}{\partial n} ds, \qquad (2.9)$$

$$\alpha H + qx = \frac{1}{2}(\varphi_+ e^{-mz} - \varphi_- e^{mz}) = q \int_{(s)} \xi \operatorname{ch} m(x - \xi) \frac{\partial g}{\partial n} ds. \qquad (2.10)$$

The value $\frac{\partial g}{\partial n}$ entering into these formulas admits the following simple /1723 physical interpretation in terms of the electrostatic problem. We will assume that the fluid is removed from the pipe which we have investigated, the walls have been made conductive and within these walls there is an infinitely long filament with a charge having the linear density $e^0 = -\cos mz$; the charge varies sinusoidally along the length of the filament; the filament passes through the point $P(x, y, 0)$, parallel to the axis of the pipe (z-axis). Under the influence of this charge induced charges appear on the walls of the pipe. These induced charges have the surface density $\bar{\sigma}_P = \sigma_P(s) \cos mz$, where $\sigma_P(s)$ is dependent only on the coordinates ξ, η of the considered point on the wall of the pipe, i.e., only on the position on the contour s of its projection Q on the plane z = 0, and also on the coordinates x, y of the point P. In this case we have the relationship

$$\sigma_P(s)\Big|_Q = \frac{\partial g}{\partial n}\Big|_Q, \tag{2.11}$$

where $\frac{\partial g}{\partial n}\Big|_Q$ applies to this same point $Q(\xi, \eta)$ of the contour s as $\sigma_P(s) = \sigma_P(Q)$.[1]

The density $\sigma_P(Q)$ satisfies the following integral equation

$$\sigma_P(Q) = \frac{mK_1(mPQ)}{\pi}\cos\theta_{PQ} - \\ - \frac{m}{\pi}\int_{(s)} \sigma_P(A) K_1(mAQ)\cos\theta_{AQ}\, ds_A, \tag{2.12}$$

where $K_1(z) = -K_0'(z)$, Q and A are fixed and variable points on the contour s; θ_{MQ} is the angle between the external normal to s at the point Q and the vector MQ, where M is some point in the region; the integral is applied along the contour s (figure 1) and ds_A denotes an element of arc of the contour at the point A.

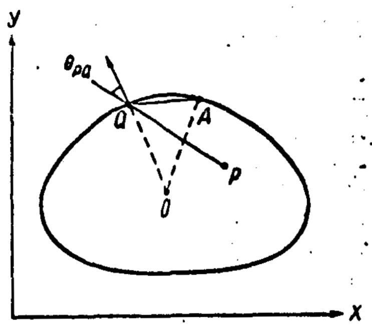

Figure 1

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[1]The explanation of this assertion and the derivation of integral equation (2.12) for the density $\sigma_P(Q)$ is given in the Appendix at the end of the paper.

If $\sigma_P(Q)$ is found from this integral equation, by substituting it in place of $\frac{\partial g}{\partial n}$ into equations (2.9) and (2.10), we obtain the solution of the formulated problem. In this paper we are concerned, as indicated above, with the case of large Hartmann numbers and we now will demonstrate that in this case an approximate solution of equation (2.12) can be found with rather general assumptions concerning the configuration of the contour s, provided we limit ourselves to a consideration of such points P(x, y) within the contour for which $ml \gg 1$, where l is the shortest distance from this point to the contour.

In actuality, for the solution of equation (2.12), in this case, we will attempt to apply the successive approximations method; as the first approximation we select the value

$$\sigma_P^{(1)}(Q) = \frac{mK_1(mPQ)}{\pi} \cos\theta_{PQ}, \tag{2.13}$$

obtained by dropping the integral term in (2.12). Since, in accordance with the condition, we have $mPQ \gg 1$, it is possible to use the asymptotic representation

$$K_1(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z},$$

which gives

$$\sigma_P^{(1)}(Q) \simeq \sqrt{\frac{m}{2\pi PQ}} e^{-mPQ} \cos\theta_{PQ}. \tag{2.14}$$

In order to obtain a second approximation $\sigma_P^{(2)}$, we introduce the value (2.14) $\sigma_P^{(1)}$ into the right-hand side of equation (2.12) /1724

$$\sigma_P^{(2)}(Q) = \sqrt{\frac{m}{2\pi PQ}} e^{-mPQ} \cos\theta_{PQ} - \frac{1}{\sqrt{2}} \left(\frac{m}{\pi}\right)^{3/2} \int_{(s)} \frac{e^{-mPA}}{\sqrt{PA}} K_1(mAQ) \times \times \cos\theta_{AQ} \cos\theta_{PA} ds_A. \tag{2.15}$$

It can be seen directly from this formula that for sufficiently large values m no appreciable contribution to the value of the integral is given except by the part of the arc of the contour directly adjacent to the point Q. In actuality, as soon as we withdraw from Q by such a distance AQ that $mAQ \gg 1$ it is possible to assume

$$K_1(mAQ) = \sqrt{\frac{m}{2\pi AQ}}\, e^{-mAQ},$$

so that the product $e^{-mPA}K_1(mAQ)$ is proportional to $e^{-m(PA + AQ)}$; here in the exponent we have the value $m(PA + AQ)$, where PA + AQ gives the length of broken PAQ, the greater exceeding the value PQ entering into the exponent in the first term to the right in equation (2.15), the farther we depart from the point Q.[1] Therefore, for sufficiently large m it is necessary in the asymptotic computation of the integral entering into (2.15) to limit ourselves solely to the immediate neighborhood of the point Q.

We now will make the corresponding computations, assuming for simplicity that Q is not an angular point of the contour. After denoting the radius of curvature of the arc of the contour at the point Q by ρ, the center of curvature by O, and introducing as an independent variable in the integration the length $AQ = \zeta$ of the chord connecting the points A and Q and assuming $PQ = a$, $\angle AOQ = 2\beta$, we obtain, assuming that $\zeta \ll a$ and $\zeta \ll 2\rho$, the following relations, true in contiguity to Q, to the right of it (see figure 1):

[1]At least in a case when the contour is convex, as we will assume hereafter as a simplification. A generalization of the derivation for more general cases could be accomplished without special difficulties.

$$\cos\theta_{AQ} = \sin\beta = \frac{\zeta}{2\rho}, \tag{2.16}$$

$$\begin{gathered} PA = \sqrt{a^2 - 2a\zeta\cos\left(\frac{\pi}{2} - \theta_{PQ} - \beta\right) + \zeta^2} = \\ = a\sqrt{1 - 2\frac{\zeta}{a}\left[\cos\theta_{PQ}\frac{\zeta}{2\rho} + \sin\theta_{PQ}\sqrt{1 - \frac{\zeta^2}{4\rho^2}}\right] + \left(\frac{\zeta}{a}\right)^2} = \\ = a - \zeta\sin\theta_{PQ} + \ldots \end{gathered} \tag{2.17}$$

$$ds_A = 2\rho d\beta = \frac{d\zeta}{\sqrt{1 - \frac{\zeta^2}{4\rho^2}}} \cong d\zeta. \tag{2.18}$$

For points A situated to the left of Q the equations (2.16) and (2.18) remain applicable, but (2.17) is replaced by PA = a + ζ sin θ_{PQ} + ...

In the asymptotic computation of the integral entering into (2.15) it is possible for sufficiently large m to approximately replace it by

$$\begin{gathered} \int_{(\epsilon)} \frac{e^{-mPA}}{\sqrt{PA}} K_1(mAQ)\cos\theta_{AQ}\cos\theta_{PA}\,ds_A \approx \\ \approx \frac{\cos\theta_{PQ}}{\sqrt{PQ}}\left[\int_0^{\epsilon} e^{-m(a-\zeta\sin\theta_{PQ})}K_1(m\zeta)\frac{\zeta}{2\rho}\,d\zeta + \int_0^{\epsilon} e^{-m(a+\zeta\sin\theta_{PQ})}K_1(m\zeta)\frac{\zeta}{2\rho}\,d\zeta\right] = \\ = \frac{\cos\theta_{PQ}}{m^2\rho\sqrt{PQ}} e^{-mPQ}\int_0^{m\epsilon} xK_1(x)\,\mathrm{ch}\,(x\sin\theta_{PQ})\,dx \simeq \\ \simeq \frac{\sqrt{2\pi}}{m^{5/2}\rho}\sigma_P^{(1)}(Q)\int_0^{\infty} xK_1(x)\,\mathrm{ch}\,(x\sin\theta_{PQ})\,dx = \frac{\pi^{3/2}}{\sqrt{2}\,m^{5/2}\rho\cos^3\theta_{PQ}}\sigma_P^{(1)}(Q), \end{gathered} \tag{2.19}$$

and it is assumed that $\epsilon \ll a$ and $\epsilon \ll 2\rho$, but $m\epsilon \gg 1$ and the contributions to the integral from arcs adjacent to Q on the right and left are taken into account separately and the slowly changing variable factor /1725

$$\cos\theta_{PA}\Big/\sqrt{PA}\,\sqrt{1 - \frac{\zeta^2}{4\rho^2}}$$

is replaced by its value at the point Q.

By introducing the derived value (2.19) into equation (2.15) we find that within the limits of the accuracy of our computations, we have the relation

$$\sigma_P^{(2)}(Q) = \sigma_P^{(1)}(Q)\left[1 - \frac{1}{2m\rho\cos^3\theta_{PQ}}\right]. \tag{2.20}$$

Since $m\rho \gg 1$ and since for an inner point of a convex contour not too close to its boundary $\cos\theta_{PQ}$ does not become equal to zero,[1] it follows from (2.20) that for sufficiently large values m $\sigma_P^{(2)}(Q)$ differs as little as desired from $\sigma_P^{(1)}(Q)$, i.e., in this case, the integral term in equation (2.12) gives only a relatively small contribution to the value $\sigma_P(Q)$, which in the future, depending on the desired accuracy, we will assume equal to $\sigma_P^{(2)}(Q)$ or even simply equal to $\sigma_P^{(1)}(Q)$.

Comment. As an example we will consider the case of a circular contour s and we will consider the point P coinciding with the center of the circle. Assuming the radius of the circular contour is equal to R, we find, by using (2.20) and (2.13) and noting that in this case $PQ = R$, $\theta_{PQ} = 0$,

$$\sigma_P^{(2)}(Q) = \frac{mK_1(mR)}{\pi}\left(1 - \frac{1}{2mR}\right). \tag{2.21}$$

This approximate formula for $\sigma_P(Q)$ can be compared with a precise formula, since in this case the precise Green's function is found easily and is equal to[2]

[1]Except the case when the contour has a very elongated configuration; then $\cos\theta_{PQ}$ can be as close to zero as desired.

[2]Because (2.22) obviously satisfies equation (2.5), it becomes equal to zero when $r = R$ and when $r = 0$ has a singularity of the required type (compare equation (2.7)).

$$g(P,\ Q) = -\frac{1}{2\pi}\left[K_0(mr) - K_0(mR)\,\frac{I_0(mr)}{I_0(mR)}\right], \tag{2.22}$$

where r is the distance of the point Q from the center P of the circle. According to (2.11), we now obtain the precise value $\sigma_P(Q)$, to wit:

$$\sigma_P(Q) = \left.\frac{\partial g}{\partial r}\right|_{r=R} = \frac{1}{2\pi R I_0(mR)}. \tag{2.23}$$

Taking into account that when $z \gg 1$ the asymptotic expressions for $I_0(z)$ and $K_1(z)$ have the form

$$\left.\begin{aligned} I_0(z) &\doteq \frac{e^z}{\sqrt{2\pi z}}\left[1 + \frac{1}{8z} + \frac{9}{2(8z)^2} + \cdots\right], \\ K_1(z) &= \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1 + \frac{3}{8z} - \frac{15}{2(8z)^2} + \cdots\right], \end{aligned}\right\} \tag{2.24}$$

we find that /1726

$$\frac{\sigma_P^{(2)}(Q)}{\sigma_P(Q)} = 1 - \frac{1}{(2mR)^2} + \cdots, \tag{2.25}$$

whereas for the first approximation we have

$$\frac{\sigma_P^{(1)}(Q)}{\sigma_P(Q)} = 1 + \frac{1}{2mR} + \cdots, \tag{2.26}$$

that is, the introduction of a correction factor in equation (2.20) appreciably improves the result.

3. We now will proceed to determination of v and H. Using (2.11), we will rewrite equations (2.9)-(2.10) so that

$$v = -q\int_{(s)} \xi\, \mathrm{sh}\, m(x-\xi)\, \sigma_P(Q)\, ds, \tag{3.1}$$

$$\alpha H + qx = q\int_{(s)} \xi\, \mathrm{ch}\, m(x-\xi)\, \sigma_P(Q)\, ds, \tag{3.2}$$

where x, y are the coordinates of that point P at which the values v and H are analyzed, and ξ, η are the coordinates of the point Q of the contour s to which the element ds of the arc applies.

For sufficiently large values of the Hartmann number it is possible to assume approximately in (3.1)-(3.2) that $\sigma_P(Q) = \sigma_P^{(1)}(Q)$, which gives

$$v = -\frac{mq}{\pi}\int_{(s)} \xi \operatorname{sh} m(x-\xi) K_1(mPQ)\cos\theta_{PQ}ds, \tag{3.3}$$

$$\alpha H + qx = \frac{mq}{\pi}\int_{(s)} \xi \operatorname{ch} m(x-\xi) K_1(mPQ)\cos\theta_{PQ}ds. \tag{3.4}$$

Taking into account that

$$PQ = \sqrt{(x-\xi)^2+(y-\eta)^2} \geqslant |x-\xi|$$

and that when $mPQ \gg 1$ the products

$$m(x-\xi)K_1(mPQ) \text{ and } \operatorname{ch} m(x-\xi)K_1(mPQ)$$

have the order of magnitude

$$\frac{e^{-m\left[\sqrt{(x-\xi)^2+(y-\eta)^2}-|x-\xi|\right]}}{\sqrt{mPQ}},$$

we see that an appreciable contribution to the values of the integrals comes only from the segments of the contour s adjacent to those points Q_1 and Q_2 at which it is intersected by the straight line $\eta = y$, parallel to the external magnetic field H^0 (figure 2). Denoting by ξ_1 and ξ_2 the ξ-th coordinates of these points, and in accordance with figure 2, $x > \xi_1$ and $x < \xi_2$, noting that in contiguity with Q_1 and Q_2 it can be assumed approximately that

$$K_1(mPQ) = \sqrt{\frac{\pi}{2m|x-\xi_i|}}\, e^{-m|x-\xi_i| - \frac{m(y-\eta)^2}{2|x-\xi_i|}}, \quad i = 1, 2$$

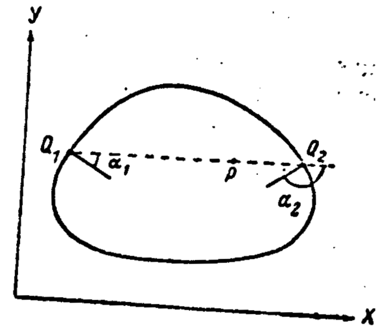

Figure 2

and substituting this value into (3.3) we find, by replacing the slowly variable factors $\xi \cos \theta_{PQ}$ by their values at the points Q_1 and Q_2 and removing them from the integral signs[1]

$$v \simeq \frac{q}{2^{3/2}} \sqrt{\frac{m}{\pi}} \sum_{i=1,2} (-1)^i \frac{\xi_i \cos \theta_{PQ_i}}{\sqrt{|x-\xi_i|}} \int_{-\infty}^{\infty} e^{-\frac{m(y-\eta)^2}{2|x-\xi_i|}} \frac{d\eta}{\cos \theta_{PQ_i}} = \frac{q}{2}(\xi_2 - \xi_1) = \frac{p_0}{4\eta m} Q_1 Q_2. \tag{3.5}$$

Here, as before, it is assumed that $\cos \theta_{PQi} \neq 0$, the curvature of the curve in the segments of integration in contiguity with Q_1 and Q_2 is neglected, i.e., it is assumed that $d\eta = ds \cos \theta_{PQi}$, and integration, as usual in such cases, is carried out in the entire interval /1727

$$-\infty \leqslant \eta \leqslant \infty.$$

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[1] The factor $(-1)^i$ appears under the sign of the sum in (3.5) due to the fact that in contiguity with

$$Q_1 \ \mathrm{sh}\, m(x-\xi) \simeq \frac{1}{2} e^{m(x-\xi)} = \frac{1}{2} e^{m|x-\xi|},$$

and in contiguity with

$$Q_2 \ \mathrm{sh}\, m(x-\xi) \simeq \simeq -\frac{1}{2} e^{m|x-\xi|}.$$

Thus, the velocity v is determined only by the length of the chord Q_1Q_2, corresponding to the coordinate y of the point P and is not dependent on x ("layered movement").

Similarly we obtain

$$\alpha H + qx \simeq \frac{q}{2}(\xi_1 + \xi_2), \tag{3.6}$$

which also can be written as

$$H \simeq \frac{2\pi P^0}{\mu H^0}[(\xi_2 - x) - (x - \xi_1)], \tag{3.7}$$

where $(x - \xi_1)$ and $(\xi_2 - x)$ are the distances from the considered point P(x, y) to the points $Q_1(\xi_1, y)$ and $Q_2(\xi_2, y)$ of the contour and it is taken into account that

$$\frac{q}{\alpha} = \frac{4\pi P^0}{\mu H^0}.$$

It follows from the above, in particular, that if each chord Q_1Q_2 is divided in half and a curve is passed through the points of separation the field H on this curve everywhere is equal to zero.[1] It divides the region within the contour s in which the field H becomes equal to zero into two parts, in each of which the currents circulate independently, without flowing from one part to the other. It also can be seen from (3.7) that

$$\frac{\mu H^0}{4\pi}\frac{\partial H}{\partial x} = \frac{1}{c}[jB]_x = -P^0,$$

[1]If v is used to denote the normal to this curve at some point along it, and τ is used to denote the tangent, it follows from the formula

$$\frac{4\pi}{c} j_\nu = [\operatorname{grad} H, \mathbf{i}_z]_\nu = \pm\frac{\partial H}{\partial \tau} = 0$$

that this curve is not intersected by the flow lines.

where

$$\mathbf{j} = \frac{c}{4\pi} \operatorname{rot} H = \frac{c}{4\pi} [\operatorname{grad} H, \mathbf{i}_z]$$

is current density in the fluid and

$$\mathbf{B} = \mu (\mathbf{i}_x H^0 + \mathbf{i}_z H)$$

is the vector of magnetic induction, i.e., that in the entire region within the pipe for which equations (3.5) and (3.7) were derived, the pressure drop is balanced by electrodynamic forces and the forces of viscous friction are negligibly small.

4. In section 3 we found general expressions (3.5) and (3.7) for v and H, assuming that $mPQ \gg 1$, i.e., that the point P at which the flow is studied is not located in the immediate neighborhood of the walls of the pipe.

We now will proceed to a study of phenomena in the layer next to the wall. In this case it would be possible to obtain the results desired directly by computation of the integrals entering into equations (3.1)-(3.2). However, we will select a different approach which is less rigorous but, on the other hand, is very simple and graphic. We proceed on the basis of equation (2.2) and from the already derived solutions (3.5) and (3.7), which we will assume are approximately correct up to the boundary (arbitrarily selected) of the layer next to the wall, if the latter is approached from within the pipe. It will be shown below that the results are not dependent (within the limits of accuracy of the problem considered) on the selection of the position of this boundary, as should be the case.

We now will consider phenomena in the layer next to the wall in contiguity with these same points Q_i, $i = 1, 2$, of the contour which were discussed in section 3.[1] We now introduce a local system of rectangular coordinates n, τ /1728
with origin at the point Q_i; the n-axis is directed along the normal to the contour within the pipe and the τ-axis is directed along the tangent; n, τ and i form a right-handed trihedral. Next, denoting by α_i the angle between the directions n and $i_x \| H^0$, at the point Q_i, which we consider different from $\frac{\pi}{2}$, we will have

$$\begin{aligned} \Delta w_{\pm} \pm 2m \frac{\partial w_{\pm}}{\partial x} = \frac{\partial^2 w_{\pm}}{\partial n^2} + \frac{\partial^2 w_{\pm}}{\partial \tau^2} \pm \\ \pm 2m \left(\frac{\partial w_{\pm}}{\partial n} \cos \alpha_i + \frac{\partial w_{\pm}}{\partial \tau} \sin \alpha_i \right) = 0. \end{aligned} \tag{4.1}$$

Since for sufficiently large values of the Hartmann number the thickness h of the layer next to the wall is small in comparison with the radius of curvature ρ of the contour s at the point Q_i, we can approximately neglect the curvature of the layer in contiguity with Q_i and also neglect the derivatives of τ in (4.1) in comparison with the derivatives of n. Thus, equation (4.1) assumes the form

$$\frac{\partial^2 w_{\pm}}{\partial n^2} \pm 2m \cos \alpha_i \frac{\partial w_{\pm}}{\partial n} = 0, \tag{4.2}$$

and when $n = 0$ the boundary conditions (2.3) should be satisfied. The latter in this case should be written simply as

$$w_{\pm}|_{n=0} = \pm q \xi_i, \quad i = 1, 2. \tag{4.3}$$

[1]That is, in essence at any points of the contour.

With respect to the second boundary condition, we will assume that when n = h, i.e., at the outer boundary of the layer next to the wall, v and H attain values determined by the equations (3.5) and (3.6). If we neglect the value h in comparison with ξ_i this gives

$$w_{\pm}|_{n=h} = [v \pm (\alpha H + qx)]|_{n=h} = \begin{cases} q\xi_2 & \text{for the sign "+".} \\ -q\xi_1 & \text{for the sign "-".} \end{cases} \quad (4.4)$$

Thus, the boundary conditions from the direction of the point Q_1 will be

$$w_+|_{n=0} = q\xi_1;\ w_+|_{n=h} = q\xi_2;\ w_-|_{n=0} = w_-|_{n=h} = -q\xi_1, \quad (4.5)$$

and from the direction Q_2

$$w_+|_{n=0} = w_+|_{n=h} = q\xi_2;\ w_-|_{n=0} = -q\xi_2;\ w_-|_{n=h} = -q\xi_1. \quad (4.6)$$

Since it follows from (4.2) that

$$(4.7)$$

where $A_{\pm}$, $B_{\pm}$ are constants and we have $\cos\alpha_1 > 0$ and $\cos\alpha_2 < 0$, and provided h is such that $2m|\cos\alpha_1|h \gg 1$, such as in contiguity with Q_1, we will have (with this same accuracy)

$$w_- = -q\xi_1 = \text{const};\ w_+ = q[\xi_2 + (\xi_1 - \xi_2)e^{-2mn\cos\alpha_1}], \quad (4.8)$$

and in contiguity with Q_2

$$w_+ = q\xi_2 = \text{const};\ w_- = -q[\xi_1 + (\xi_2 - \xi_1)e^{2mn\cos\alpha_2}]. \quad (4.9)$$

Hence, for v and H we obtain the equations

$$v = \frac{q}{2}(\xi_2 - \xi_1)(1 - e^{-2mn\cos\alpha_1}), \tag{4.10}$$

$$H = \frac{2\pi P^0}{\mu H^0}(\xi_2 - \xi_1)(1 - e^{-2mn\cos\alpha_1}); \tag{4.11}$$

which determine the change of v and H with movement from the point Q_1 within /1729
the pipe along the normal to the contour.[1]

Similar equations are derived for the point Q_2.

We note that with movement from Q_i along the normal through the layer next to the wall the magnetic field H changes by the value

$$\Delta H = \pm \frac{2\pi P^0}{\mu H^0}(\xi_2 - \xi_1),$$

which indicates the presence of currents

$$I = \pm c\ P^0(\xi_2 - \xi_1)/2\ \mu\ H^0$$

(per unit length of the axis of the pipe), flowing in thes $w_\pm = A_\pm + B_\pm e^{\mp 2m\cos\alpha_i \cdot n}$,
along the contour and in the backward direction.

5. We also will consider a case of a rectangular contour with the sides l and d in the direction of the field H^0 and perpendicular to it. The precise Green's function for this case, given and used for solution of the considered problem in our study cited as ref. 5, section 5, is derived easily by the reflection method and is equal to

$$\begin{aligned} g(P, Q) \equiv g(x, y, \xi, \eta) = -\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\{K_0[m\sqrt{(x_r-\xi)^2+(y_s-\eta)^2}] - \\ -K_0\left[m\sqrt{(x_r-\xi)^2+(y'_s-\eta)^2}\right] - K_0\left[m\sqrt{(x'_r-\xi)^2+(y_s-\eta)^2}\right] + \\ +K_0\left[m\sqrt{(x'_r-\xi)^2+(y'_s-\eta)^2}\right]\}, \end{aligned} \tag{5.1}$$

[1]We note that $n\cos\alpha_1 = x - \xi_1$.

where

$$x_r = 2rl + x,\ x'_r = 2rl - x,\ y_s = 2sd + y,\ y'_s = 2sd - y; \tag{5.2}$$

here the origin of coordinates is situated in the lower left corner of the rectangle and the x- and y-axes are directed accordingly to the right along its lower side and upward along its left side.

If the double series (5.1) converges more rapidly, the larger is the value of the Hartmann number. Therefore, if $ml \gg 1$ and $md \gg 1$, in order to obtain an approximate solution even with a high degree of accuracy it is sufficient to limit ourselves, when substituting g into equations (2.9)-(2.10), to only a few terms of the series (5.1). For example, if it is necessary to find v at the point P(x, y), not lying within the layer next to the wall, i.e., such that $mb \gg 1$ (where b is the shortest distance from P to the nearest wall), the only terms making a substantial contribution to the value v(x, y), determined using equation (2.9), are

$$\frac{1}{2\pi}\left\{K_0\left[m\sqrt{(x-\xi)^2+(y-\eta)^2}\right]-K_0\left[m\sqrt{(2l-x-\xi)^2+(y-\eta)^2}\right]\right\},$$

which give

$$\begin{aligned} v(x,\ y) &\simeq \frac{qlm}{\pi}(x-l)\,\mathrm{sh}\, m(x-l)\int_0^d \frac{K_1\left[m\sqrt{(l-x)^2+(y-\eta)^2}\right]}{\sqrt{(l-x)^2+(y-\eta)^2}}\,d\eta \simeq \\ &\simeq \frac{qlm}{\pi}(l-x)\,\mathrm{sh}\, m(l-x)\int_{-\infty}^{\infty} \frac{K_1\left(m\sqrt{(l-x)^2+z^2}\right)}{\sqrt{(l-x)^2+z^2}}\,dz = \\ &= ql\,\mathrm{sh}\, m(l-x)\, e^{-m(l-x)} \simeq \frac{ql}{2} \end{aligned} \tag{5.3}$$

in full accordance with formula (3.5) of the general theory.

If the point P(x, y) is moved toward the wall x = 0 to such an extent that the inequality $mx \gg 1$ no longer is satisfied, but as before we would have $my \gg 1$ and $m(d - y) \gg 1$, it also would be necessary to take into account terms corresponding to the reflection P'(-x, y) of the point P(x, y) in the direction x = 0 of the rectangle, and also the reflection $P''(2l + x, y)$ of the point P' in the direction $x = l$, i.e., the terms

$$\frac{1}{2\pi}\left(K_0\left(m\sqrt{(x+\xi)^2+(y-\eta)^2}\right)-K_0\left(m\sqrt{(2l+x-\xi)^2+(y-\eta)^2}\right)\right).$$

With these terms taken into account, formula (5.3) would be replaced, as can be confirmed easily, by /1730

$$v(x, y)=\frac{ql}{2}(1-e^{-2mx}), \tag{5.4}$$

which corresponds fully to the equation (4.10) of the general theory, in which for this particular case it must be assumed that $\xi_2 - \xi_1 = l$, $\alpha_1 = 0$ and $n = x$.

In a similar way it would be possible to consider any other positions of the point P, e.g., when it is located in the immediate neighborhood of one of the walls $y = 0$ or $y = d$ or very near one of the angles. By increasing the number of terms used in the series (5.1) in case of necessity it always is possible to obtain a solution of the problem with any desired degree of accuracy.

6. We note in conclusion that although in this paper we have been concerned for the most part with the case of large Hartmann numbers the method discussed, involving the reduction of the general problem formulated at the beginning of the paper to the solution of the integral equation (2.12) for the density $\sigma_P(Q)$ and the subsequent quadratures (2.9)-(2.10), has a fully general character. The simplification, introducing the assumption of a large value of the Hartmann number, was for no other purpose than to make it possible to indicate the rapidly converging process of successive approximations for finding

$\sigma P(Q)$ and these approximations had a simple form.[1] For small and intermediate values of the Hartmann number this process can be unsuitable or disadvantageous, and then the solution of the problem requires the use of numerical or some other methods of approximate solution of equation (2.12). We note also that in this case it is unimportant whether the contour is simply or multiply-connected, convex or not, etc., because the principal equation (2.12) and equations (2.9)-(2.10) remain true in all cases, provided that for a multiply-connected region s is understood as the set of contours limiting this region.

Appendix

Explanation of the Electrostatic Analogy and Derivation of the Integral Equation (2.12)

We will consider here the internal electrostatic problem for an infinite linear pipe with conducting walls, within which there is an infinitely long filament with a charge having the density $e^0 = e \cos mz$ (e = const); the charge varies sinusoidally along its length; the filament passes through the point $P(x, y, 0)$, parallel to the axis of the pipe (z-axis). At the point $M(\xi, \eta, z)$ within the pipe such a filament creates an electrostatic field with the potential

$$\Phi_1 = e\int_{-\infty}^{\infty} \frac{\cos m\zeta\, d\zeta}{\sqrt{r^2 + (z-\zeta)^2}} = 2eK_0(mr)\cos mz \equiv \varphi_1 \cos mz,$$

---

[1] We have limited ourselves above to determination of the principal terms of the asymptotic formulas for $v(x, y)$ and $H(x, y)$ for large values of the Hartmann number. It is obvious that by using the same method, but making the estimates more precisely, it would be possible to obtain the succeeding terms of the asymptotic formulas as well.

where

$$r=\sqrt{(x-\xi)^2+(y-\eta)^2}.$$

This field induces on the walls of the pipe charges with the surface density $\bar{\sigma}_P = \sigma_P(N)\cos mz$, where $\sigma_P(N)$ is dependent only on the coordinates of the projection N of the considered point of the wall on the plane z = 0, i.e., on its position on the contour s and on the coordinates x, y of the point P. These /1731 charges create within the pipe a secondary field with the potential $\Phi_2 = \varphi_2(\xi, \eta)\cos mz$. It follows from the relation

$$\frac{\partial^2\Phi_2}{\partial\xi^2}+\frac{\partial^2\Phi_2}{\partial\eta^2}+\frac{\partial^2\Phi_2}{\partial z^2}=0$$

that φ_2 satisfies within s the equation

$$\frac{\partial^2\varphi_2}{\partial\xi^2}+\frac{\partial^2\varphi_2}{\partial\eta^2}-m^2\varphi_2=0,$$

i.e., an equation of the form (2.5).

If it is required further than on the contour s the total potential $\varphi = \varphi_1 + \varphi_2$ becomes equal to zero, a comparison with the conditions used for determination of the Green's function introduced in section 2 shows directly that $g(P, Q) = -\frac{\varphi}{4\pi e}$. The value of the potential φ naturally applies to the same point $Q(\xi, \eta)$ at which the Green's function g(P, Q) is considered. In this case for any point Q of the contour we obtain

$$\left.\frac{\partial g}{\partial n}\right|_Q=-\frac{1}{4\pi e}\left.\frac{\partial\varphi}{\partial n}\right|_Q=-\frac{1}{e}\sigma_P(Q),$$

it therefore can be seen that when

$$e=-1\ \left.\frac{\partial g}{\partial n}\right|_Q=\sigma_P(Q),$$

we obtain precisely equation (2.11) of the principal text of the paper.

We now will write an integral equation which should satisfy the density σ_P. We note first that the total electric field E, being the sum of the primary field E_1 = -grad Φ_1 and the secondary field E_2 = -grad Φ_2 should become equal to zero outside the pipe, since the charges σ_P induced on the walls fully connect all the lines of force of the primary field. Formulating the requirement that the component E_n of the total field normal to the contour should become equal to zero with approach of the contour s from the outside to any point Q of the contour and the E_n consists of the normal components of the primary field $(E_1)_n = \frac{\partial \Phi_1}{\partial n}$ and the secondary field $(E_2)_n$, which according to the well known theorem of potential theory is equal to the sum

$$-\frac{\partial \Phi_2}{\partial n}\Big|_Q + 2\pi\bar{\sigma}_P(Q),$$

where

$$-\frac{\partial \Phi_2}{\partial n}\Big|_Q$$

is the value of the normal component from the induced densities $\bar{\sigma}_P$ at the surface of the pipe itself, we obtain equation (2.12) precisely and it is only necessary to take into account that $\Phi_2 = \varphi_2 \cos mz$ and $\varphi_2(M) = 2\int_{(s)} \sigma_P(A) K_0(mAM) ds_A$.

The notations are the same as in (2.12).

A. F. Ioffe Physical-Technical Institute,
Academy of Sciences USSR, Leningrad

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